Brandeis University

Summer 2024

MATH 15A - Name: _______________– Linear Algebra

June 24th TIME ALLOWED: 130 MINUTES

INSTRUCTIONS TO CANDIDATES

1. There are 5 regular Questions, plus one *bonus* question.
2. There are also several *bonus* sub-questions in some questions. Those are not necessarily very challenging, rather they are to give you more chances for the exam.
3. *The last bonus question (Q6) can be challenging, be careful with spending time on it.*
4. One cheat sheet is **allowed**.
5. Show all the work!

Write your name here: ____________________________________

SCORE PAGE

1. Q1:
2. Q2:
3. Q3:
4. Q4:
5. Q5:
6. Bonus:
7. Total:

Question 1 (10 Points + 2 Bonus)

For the following 5 **True or False** questions, briefly justify your answer:

1. Given matrices $M_{n\times p}, A_{m\times n}, B_{p\times t}$, the multiplication $(AM)B$ has **size** $n \times t$.
   False. The size is $m \times t$.

2. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ has a **corresponding** $n \times m$ standard matrix A such that $T(\vec{x}) = A\vec{x}$.
   False. Should be $m \times n$.

3. Let $A = \begin{pmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} & \vec{v_4} \end{pmatrix}$ be a 4×4 matrix and let $\vec{b}$ be a vector in $\mathbb{R}^4$. If $A\vec{x} = \vec{b}$ is **consistent**, then $\vec{v_4}$ is a **linear combination of** $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{b}\}$.
   False. Cannot guarantee that x_4 *is nonzero.*

4. Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with $m \neq n$. Then T **cannot be both one-to-one and onto**.
   True.

5. For the space of polynomials with real coefficients and **no** highest degree required, it **does not** have a basis.
   False, a basis can be given as $\{1, x, x^2, \cdots, x^n, \cdots\}$. *(True if you mention in the book basis is finite.)*

(Bonus 2 Points) The subset of all invertible matrices in the vector space of $n \times n$ matrices is itself a vector space (with addition as the operation).
False. No zero vector.

Question 2 (20 Points + 5 Bonus)

Given the linear system (**Be careful with the subscripts!**):

$$\begin{cases} x_1 + x_2 + 3x_3 + 3x_4 = -2 \\ -2x_1 + x_2 = 1 \\ 2x_1 + x_3 + x_4 = 1 \end{cases} .$$

(i) Write its **coefficient matrix** A and **augmented matrix** B. (5 Points)

(ii) Compute the **(reduced) echelon form** of the augmented matrix. Circle the **pivot positions**. (10 Points)

(*Remark: not reduced echelon form is fine, whichever is easier for you for the following questions.*)

(iii) Write the **solution set** $\vec{x}$ from the result you obtained above, and name the **free variable(s)**. (5 Points)

(Bonus 5 Points) Write down a basis of the **row spaces**, **column spaces** and **null spaces** of the matrix A. What is its **rank**?

(i). We have

$$A = \begin{pmatrix} 1 & 1 & 3 & 3 \\ -2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 1 & 3 & 3 & -2 \\ -2 & 1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

(ii). Echelon form

$$\begin{pmatrix} 1 & 1 & 3 & 3 & -2 \\ 0 & 1 & 2 & 2 & -1 \\ 0 & 0 & 1 & 1 & -3 \end{pmatrix}$$

with povit positions the left $1's$.

(iii). Solution set $\vec{x} = \begin{pmatrix} 2 \\ 5 \\ -3 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$. Free variable x_4.

(Bonus). Row space basis can be $\{(1,0,0,0),(0,1,0,0),(0,0,1,1)\}$.

(It is understandable that the question is a bit misleading, so calculate for either A or B is fine.)

Column space basis $\{(1,-2,2),(1,1,0),(3,0,1)\}$.

Null space $\{(0,0,-1,1,0),(-2,-5,3,0,1)\}$. Rank is 3.

Question 3 (25 Points)

Given a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$, which can be given by the matrix

$$A = \begin{pmatrix} -3 & 0 & 1 & 2 \\ 0 & 0 & 5 & 3 \\ 5 & 4 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix}.$$

(i) Compute the **determinant** of A^2, that is, $\det(A^2)$. (10 Points)

(Hint: Find a good column/row for your co-factor expansion; then think about the properties of determinants!)

(ii) Is this matrix **invertible**? Why? (5 Points)

(iii) Given other two matrices

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & -1 & 3 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 & -1 \\ -2 & 0 & 0 \\ 1 & -1 & 3 \end{pmatrix}.$$

Compute $2B - C$. (5 Points)

(iv) Compute the multiplication BC. (5 Points)

(Hint: Show your work and you will have partial credits for your process even if you made mistakes!)

(i). Determinant is 144^2.

(ii). Invertible because determinant nonzero.

(iii). We have

$$\begin{pmatrix} 2 & 3 & 7 \\ 2 & 0 & 0 \\ 7 & -1 & 3 \end{pmatrix}.$$

(iv). We have

$$BC = \begin{pmatrix} -1 & -2 & 8 \\ 0 & 0 & 0 \\ 5 & 1 & 5 \end{pmatrix}.$$

Question 4 (15 Points + 2 Bonus)

Consider a vector space $V = \mathbb{P}^3 = \{a_3x^3 + a_2x^2 + a_1x + a_0 | a_i \in \mathbb{R}\}$, the space of all polynomials with real coefficients and with highest possible degree 3.

(i) What is the **zero vector** of this vector space? Can you give a **basis** for this space? (10 Points)

(ii) Can you give a **nonzero subspace** H of V and is not all of V? Can you check that it is **indeed a subspace** by definition? (5 Points)

(*Hint: it is enough to check it satisfies the conditions for being a subspace, instead of the 10 properties for vector spaces.*)

(Bonus 2 Points) Now identifies $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ with $p(x) = ax^3 + bx^2 + cx + d$, can you give a 4×4 matrix that **presents the linear transformation from** V **to** H **(the subspace** H **you gave in (ii))**?

(i). Zero vector is the polynomial that is always 0.

A basis can be $\{1, x, x^2, x^3\}$.

(ii). Depends.

Question 5 (30 Points + 3 Bonus)

Given a matrix
$$M = \begin{pmatrix} 5 & 6 & -6 \\ 0 & 5 & 0 \\ 0 & 6 & -1 \end{pmatrix}.$$

We want to diagonalize it in the following way.

(i) Write its **characteristic equation**, **factorize** it, and use it to compute its **eigenvalues**. For each eigenvalue, write the (algebraic) **multiplicity** of it. (15 Points)

(ii) Compute the **eigenvector(s)** for each eigenvalue. (5 Points)

(iii) Check that those eigenvectors are **linearly independent**. (5 Points)

(Hint: Remember that the eigenvectors for different eigenvalues are automatically linearly independent. So it is enough to check the ones with the same eigenvalues, if exists.)

(iv) **Construct** P **and** D, by the eigenvalues and eigenvectors for the decomposition $M = PDP^{-1}$ we want (no need to compute P^{-1} yet!). (2 Points)

(v) Compute P^{-1}, **the inverse** of P, which would finally give the decomposition $M = PDP^{-1}$. (3 Points)

(Bonus 3 Points) Compute M^k, the $k-$th power of M.

(i). Equation $(\lambda - 5)^2(\lambda + 1) = 0$.

Eigenvalues $-1, 5, 5$. Multiplicity 2 for 5 and 1 for -1.

(ii). Eigenvector $(1, 0, 1), (1, 0, 0), (0, 1, 1)$.

(iii). Routine check.

(iv). We can have
$$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

and

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

(v). The inverse of the above P is $\begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$.

(Bonus). We have

$$M^k = \begin{pmatrix} 5^k & (-1)^{k+1} + 5^k & (-1)^k - 5^k \\ 0 & 5^k & 0 \\ 0 & (-1)^{k+1} + 5^k & (-1)^k \end{pmatrix}.$$

Question (Bonus 20 Points)

Given an $n-$dimensional real vector space V with basis $v_1, v_2, \cdots, v_n$, the linear transformations from V to itself will form another vector space (e.g. elements of this vector space contains the map f that $f(x_1, x_2, x_3, \cdots, x_n)$ to $(x_2, x_1, x_3, \cdots, x_n)$, changing x_1 and x_2). This is sometimes called *the endomorphism space of* V, denoted by $\mathrm{End}(V)$. The two operations are defined as follows:

$$(f+g)(\vec{v}) = f(\vec{v}) + g(\vec{v}),$$
$$(cf)(\vec{v}) = c(f(\vec{v})),$$

where $\vec{v} \in V$, $c \in \mathbb{R}$, and $f, g \in \mathrm{End}(V)$.

(i) Can you describe this space, in the sense that can you name how the elements in this space looks like, in particular: what is the **zero vector** in this space, and what would be a nice **basis** for this space? What is the **dimension** of this space? (10 Points)

(ii) This shows us given one finite dimensional vector space, we can naturally construct another vector space that is associated to it. There can be another one: let V be as before, consider all the linear functions on the vector space V, denoted by V^* (usually called the *dual space of* V). That is, the elements are all linear functions f such that $f(\vec{v}) = c$ for some real number c. Linearity here means also that

$$(f+g)(\vec{v}) = f(\vec{v}) + g(\vec{v}) \text{ and } (cf)(\vec{v}) = c(f(\vec{v})).$$

What do you think the **zero vector** and **dimension** of this space is? Can you give a **basis**? (10 Points)

(i). Zero vector in $\mathrm{End}(V)$ is the map that map any vector in V to zero vector in V. It has dimension n^2 and a basis can be given as the entries of $n \times n$ matrix, with one entry being 1 and all the others 0.

(ii). Zero vector in V^* can be the function that takes 0 for any input $\vec{v}$.

Dimension is n and a basis can be given by functions ϵ_i such that $\epsilon_i(v_i) = 1$ and $\epsilon_i(v_j) = 0$ for $i \neq j$, where v_i are the basis of V.

END OF PAPER

Work for Question 1

Work for Question 1

Work for Question 2

Work for Question 2

Work for Question 3

Work for Question 3

Work for Question 4

Work for Question 4

Work for Question 5

Work for Question 5

Work for Bonus Question

Work for Bonus Question

Extra Paper