Brandeis University

Summer 2024

MATH 15A - Final Quiz – Linear Algebra

July 1st TIME ALLOWED: 110 MINUTES

INSTRUCTIONS TO CANDIDATES

1. There are 4 regular Questions.
2. There are also several *bonus* sub-questions in Question 4. It is **NOT** all or nothing for the whole 30 points, but 10 points for each.
3. *Be careful with spending time on the bonus.*
4. One cheat sheet is **allowed**.
5. Show all the work!

Write your name here: ______________________________

SCORE PAGE

1. Q1:
2. Q2:
3. Q3:
4. Q4:
5. Total:

Question 1 (10 Points)

For the following 5 **True or False** questions, briefly justify your answer:

1. If we have a symmetric matrix with all entries negative, then it is negative definite.

2. Given a basis of a subspace, the Gram-Schmidt process will give the same set of vectors as the orthonormal basis regardless how you start the process.

3. If a vector $\vec{v}$ is orthogonal to every element in one basis B of the subspace W, then $\vec{v}$ is orthogonal to the space W.

4. Every vector $\vec{v}$ in a vector space V can be made into a unit vector.

5. For inner product in real vector spaces, we have
$$\langle \vec{u}, \vec{v_1} + \vec{v_2} \rangle = \langle \vec{u}, \vec{v_1} \rangle + \langle \vec{u}, \vec{v_2} \rangle.$$

Question 2 (40 Points)

Given the set S containing the vectors $\vec{v_1} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} -2 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \vec{v_3} = \begin{pmatrix} 5 \\ 4 \\ 3 \\ 1 \end{pmatrix}$ in $\mathbb{R}^4$, then

(i). Compute the **length of each** of them. (6 Points)

(ii). Check that they do not form an **orthogonal set**. (4 Points)

(iii). Further, make S into an **orthonormal basis** for the subspace $W = \text{Span}\{v_1, v_2, v_3\}$ in $\mathbb{R}^4$ by performing the **Gram-Schmidt process**. (15 Points)

(iv). Find the **unit vector** in $\mathbb{R}^4$ that is **orthogonal to this subspace** W. (5 Points)

(v). Compute the **projection** of vector $v_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ onto the space W and onto **its orthogonal complement** $W^{\perp}$. (10 Points)

Question 3 (40 Points)

Given a symmetric matrix $A = \begin{pmatrix} 6 & 10 & 0 \\ 10 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix}$

(i). Given that it has eigenvalues $16, 9, -4$ respectively, compute its corresponding **eigenvectors**, then normalize them to give an **orthonormal basis**. (10 Points)

(ii). Then we can decompose A into PDP^{-1}, write out the **decomposition**. (5 Points)

(iii). Write down the "Spectral decomposition" of A, that is, decompose A into **pieces of eigenvalues** in some way. (5 Points)

(iv). A symmetric matrix A can associate to a quadratic form Q_A such that $Q_A(\vec{x}) = \vec{x}^T A\vec{x}$. Given that $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, write down the **explicit expression** of the quadratic form $Q_A(\vec{x})$. Name the **cross-product terms**. (10 Points)

(v). We know that by a change of variable, this quadratic form can have no cross-product. Hence, find the **change of variable** $\vec{y}$ and the **corresponding matrix** P such that you will obtain a quadratic form with input $\vec{y}$ with no cross-products. (5 Points)

(vi). Is this quadratic form positive definite, negative definite, or indefinite? (5 Points)

Question 4 (10 Points + 30 Bonus)

We learned that inner products can be viewed as a generalization of the dot product in $\mathbb{R}^n$. We did not talk much about inner products other than $\mathbb{R}^n$, so let us start from here.

(i). For the vector space $\mathbb{P}^n$, the space of real polynomials with highest possible degree n, pick any two vectors, denoted as $f = a_0 + a_1x + a_2x^2 + \cdots a_nx^n$ and $g = b_0 + b_1x + \cdots + b_nx^n$, there can be natural inner product

$$\langle f, g\rangle = a_0b_0 + \cdots + a_nb_n.$$

Check that this definition **satisfies the condition for being an inner product when** $n = 3$, that is, the space is $\mathbb{P}^3$. (10 Points)

(Bonus 1). Given the vector space $M_{n\times n}$, the space of $n \times n$ matrices with real entries, we ought to also define an inner product on it. If we have two square matrices A and B with entries a_{ij}, b_{ij}, where i denote the row and j denote the column as usual, we can define

$$\langle A, B\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ij}.$$

Check that **when** $n = 2$**, this satisfies the conditions for being an inner product.**

(Bonus 2). Recall in class we mentioned that, in chapter 6, we assume that we are talking about those vector spaces that have an inner product (i.e., you can define one naturally).

Indeed, for any finite dimensional vector spaces with a basis, it actually **automatically** has an inner product that one can naturally define. Think from the examples above, then consider given any finite dimensional vector space V with an orthonormal basis $e_1, e_2, \cdots, e_n$. How could you naturally **define the inner product between two vectors** $v, w \in V$**?**

Hint: Write any element $v \in V$ in terms of an orthonormal basis.

(Bonus 3). We can also view inner products in another way. Recall from the bonus question in our midterm that, for a finite dimensional vector space V, we can naturally associate a dual space V^* of it, where the elements are linear functions on V, i.e., an element in V^* is a linear function f that such $f(v) = c$ for some number c. Then recall that a natural basis for V^* would be all the functions f_i such that $f_i(v_j) = 0$ if $i \neq j$ and 1 if $i = j$, where $v_1, \cdots, v_j, \cdots, v_n$ is a basis of V. That is, any element $\vec{v_i}$ in the basis of V can be associated naturally to an element f_i in the basis of V^*.

Now recall that the dot product is actually to calculate

$$\langle \vec{v}, \vec{u} \rangle = \vec{v} \cdot \vec{u} = \vec{v}^T \vec{u},$$

and the meaning of $\vec{v}^T$ can be understood as the corresponding element of $\vec{v}$ in the dual space V^*, then as a function with input $\vec{u}$, it outputs a real number, which actually makes sense with the meaning of inner product. That is, an inner product can be viewed as a natural result of the fact that V has a dual space V^*.

Moreover usually when we write the functions taking vectors in V, we like to write it as $\langle \vec{v}, f \rangle$ instead of $f(\vec{v})$, to *show the symmetry of their relation.* Now an operator can be written as a square matrix A, meaning that $A\vec{v} = \vec{u}$ another vector. Now think about $\langle A\vec{v}, f \rangle$, then **show that**

$$\langle A\vec{v}, f \rangle = \langle \vec{v}, A^T f \rangle.$$

This naturally induces an operator on the dual space.

END OF PAPER

Work for Question 1

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Work for Question 2

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Work for Question 3

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Work for Question 4

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